Counting Convex k-gons in an Arrangement of Line Segments

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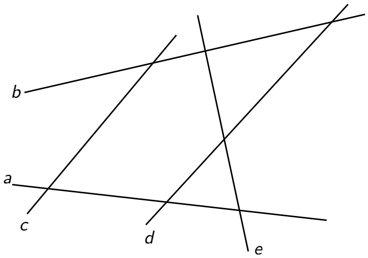
Natural generalization to convex k-gons

Problem Definition

Given: An arrangement $\mathcal{A}(S)$ of line segments S in 2-D



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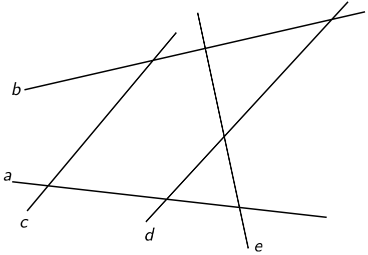

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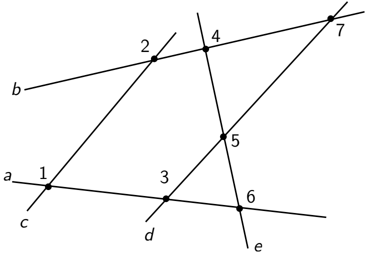

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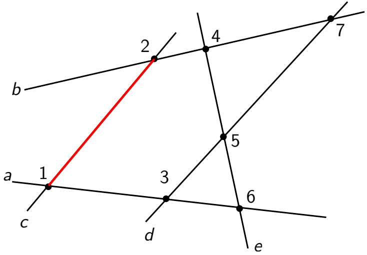

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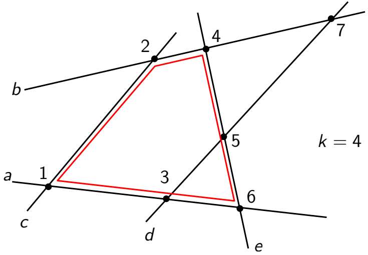

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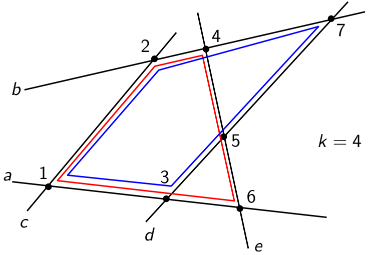

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Goal: count and report all such k-gons.

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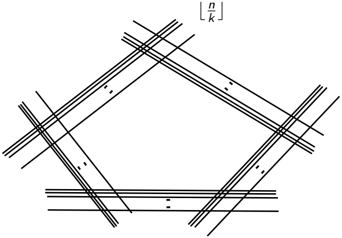

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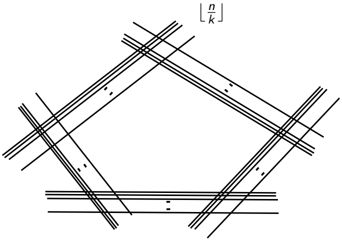
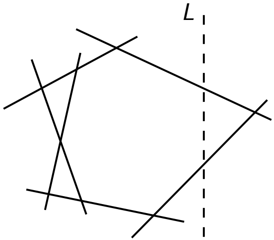


- Counting k-gons is as hard as the 3SUM problem, for $k = 3, 4$

Counting k-gons

A vertical line L intersects at most two sides of a k-gon P.



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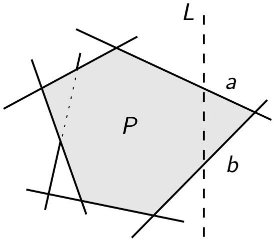

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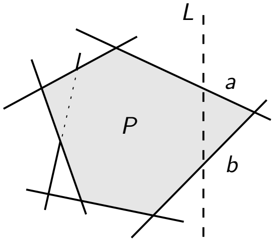

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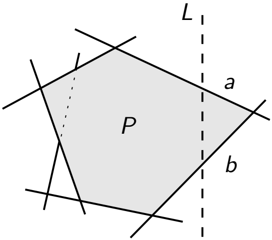

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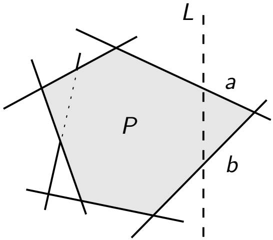

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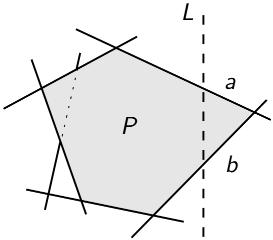

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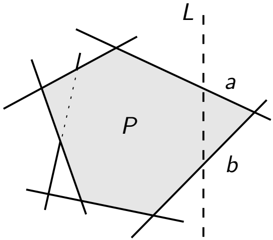

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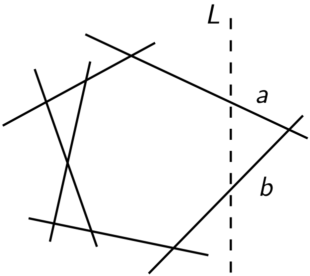
- $O(n^2)$ distinct spans (w.r.t. L) among all k-gons
- Suggests a plane sweep based algorithm, key idea:
  - Assign a k-gon intersecting L to its span
  - Update count as we sweep L across the plane

Notation



- **Open *j*-gons**: All $j \leq k$ sides start left of L

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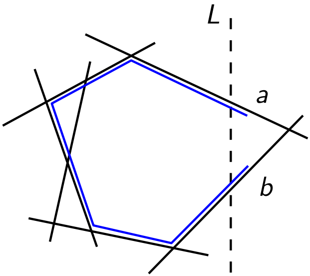

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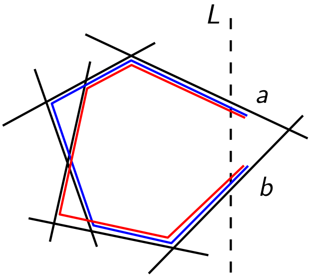

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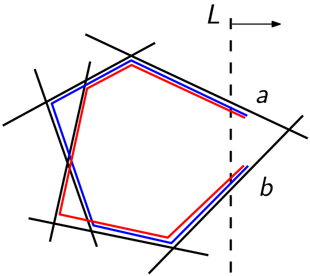

$\sigma(a, b, 5) = 2$

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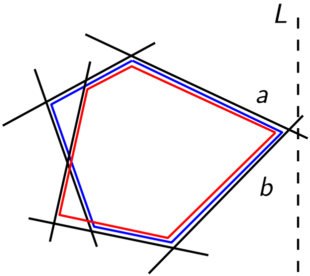

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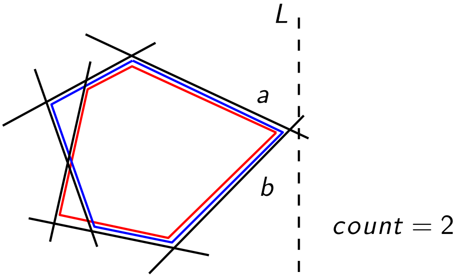

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- **Closed k-gons**: All k sides end left of L
  - *count*: number of k-gons left of L

Algorithm Steps

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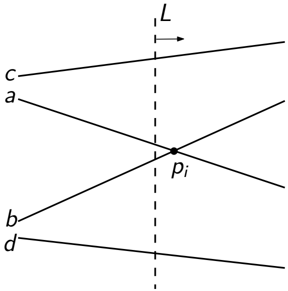
- Return *count*

Updates at intersection (a, b)



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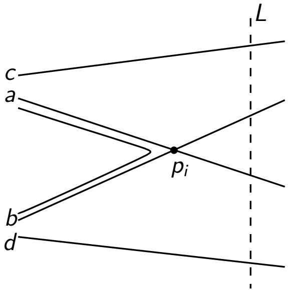

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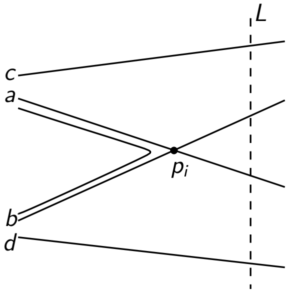

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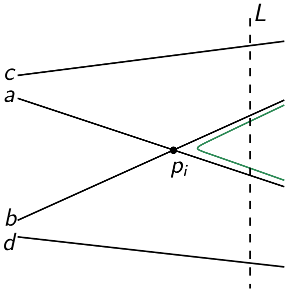

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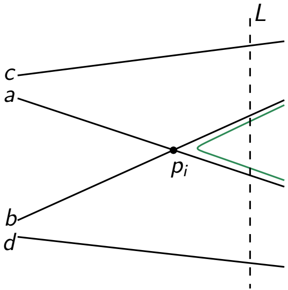

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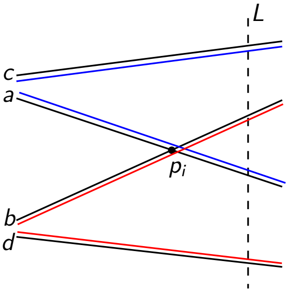

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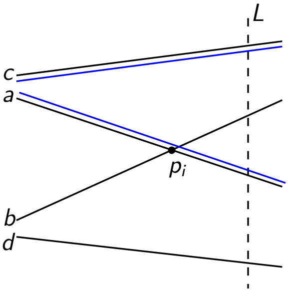

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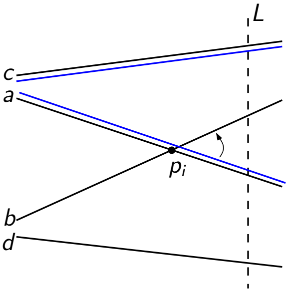

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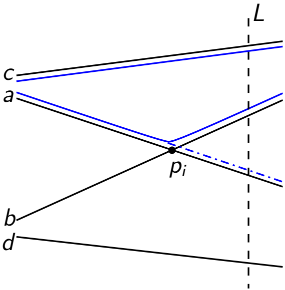

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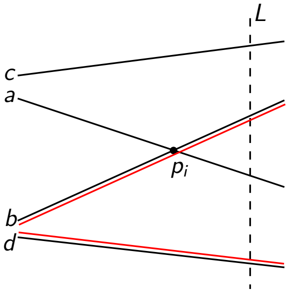

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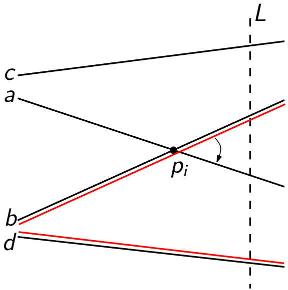

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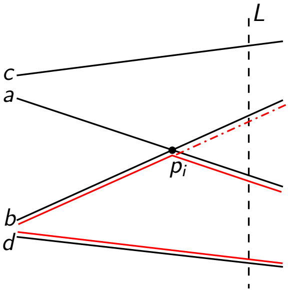

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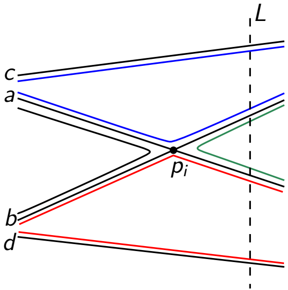

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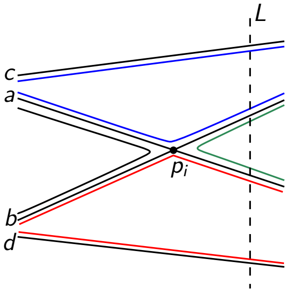

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Handles degenerate cases: apply pairwise updates collectively

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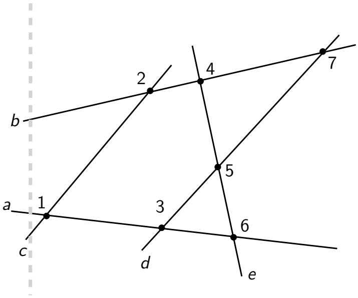

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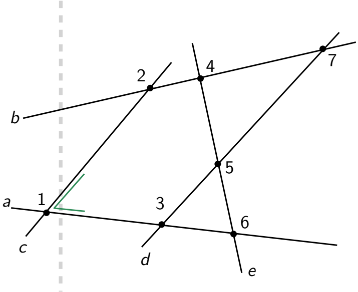

$(c, a, 2)$

Create a vertex for the new open 2-gon

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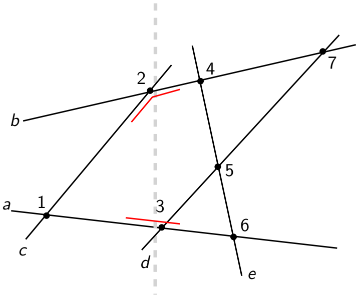

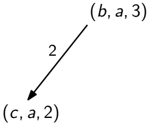

Add an edge for a j-gon growing into $j+1$-gon

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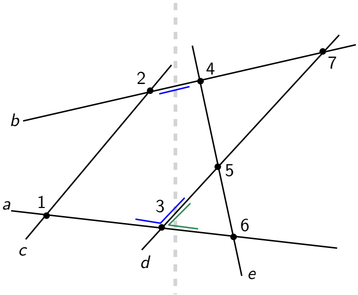

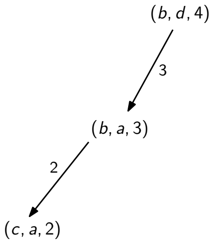

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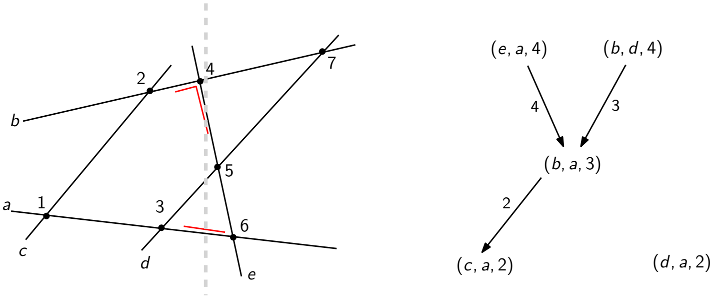

Add an edge for a j-gon growing into $j+1$-gon

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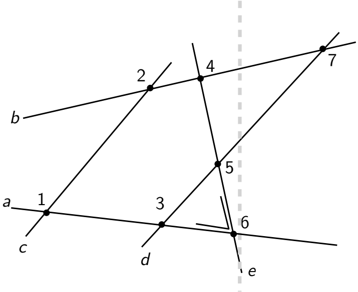

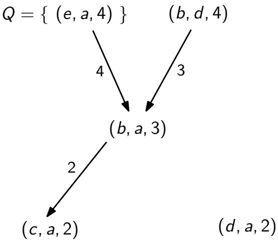

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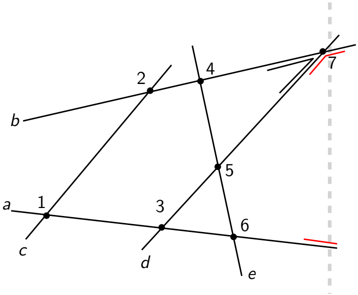

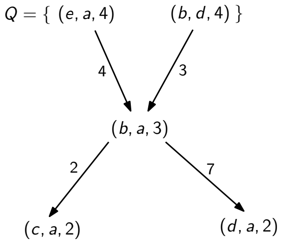

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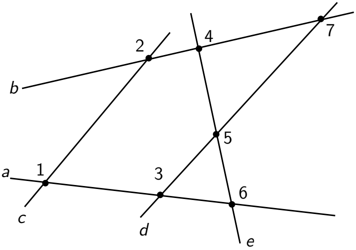

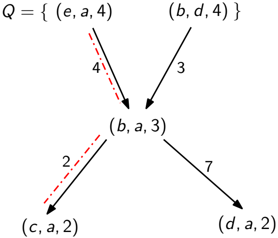

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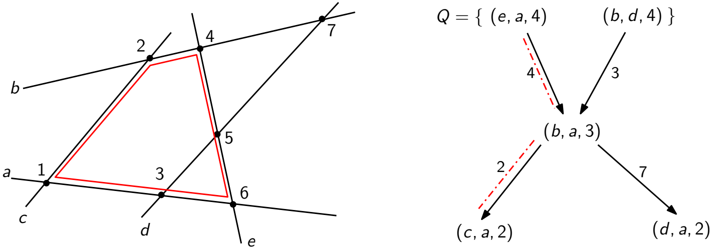

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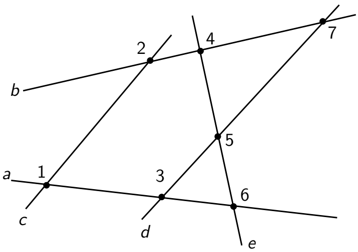

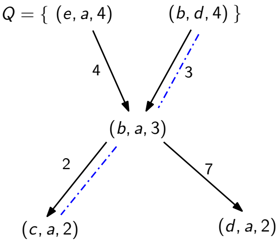

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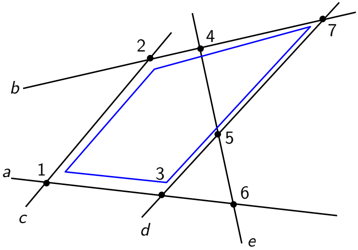

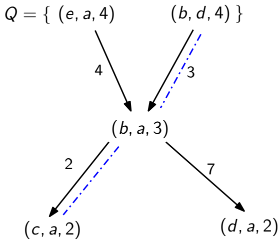

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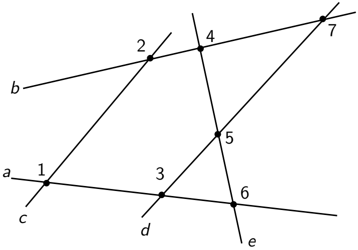

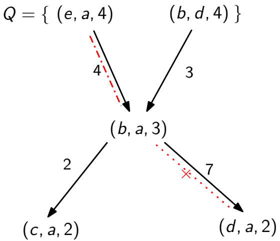

Only time respecting paths are valid k-gons

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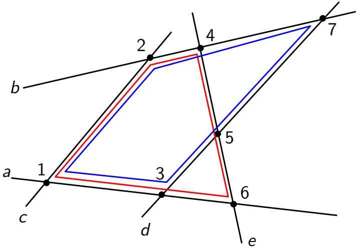

$K = \{(1, 2, 4, 6), \quad (1, 2, 3, 7)\}$

Report all k-gons in $O(|K|)$ additional time

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Reduction from POINT-ON-3-LINES problem

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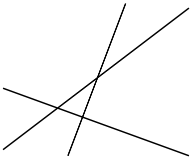

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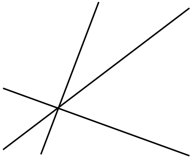

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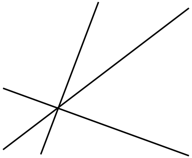

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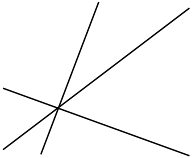

Does not extend to $k \geq 5$

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- Open question: faster algorithms?

Thanks!