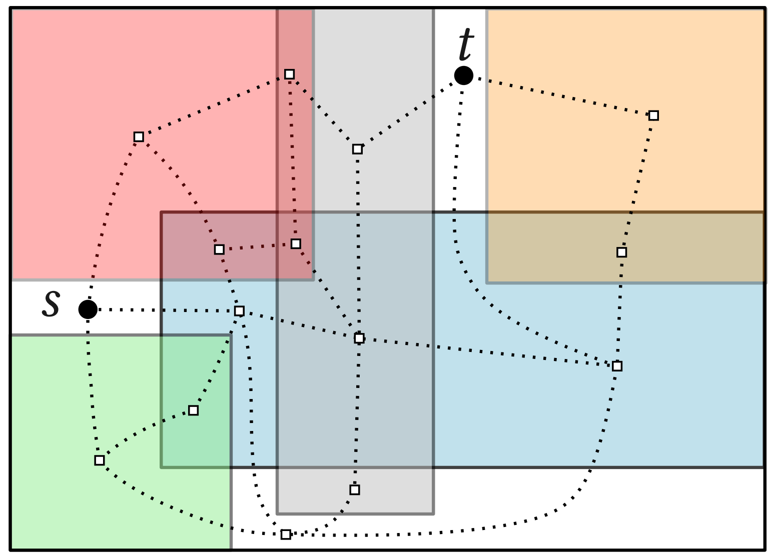
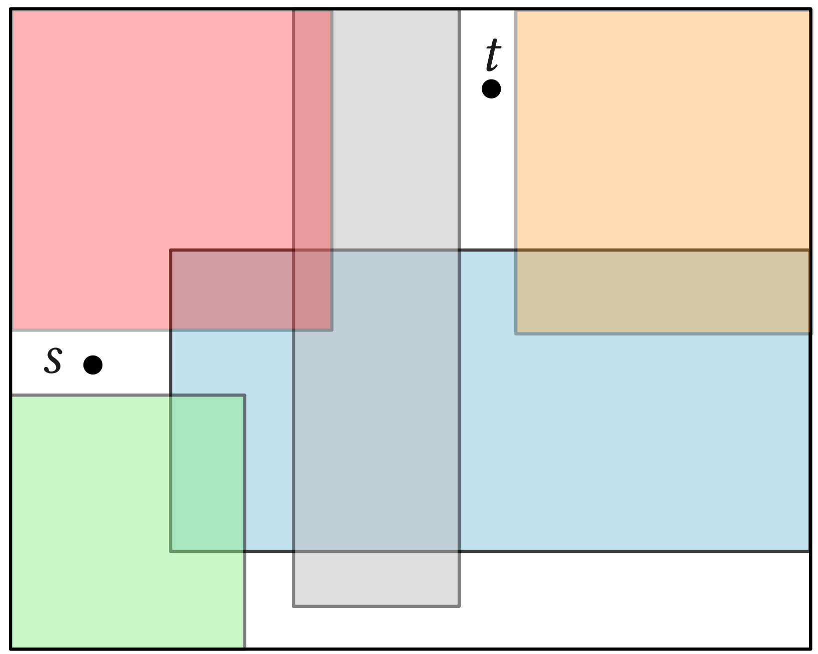


A Constant Factor Approximation for Navigating Through Connected Obstacles in the Plane

Neeraj Kumar, Daniel Lokshtanov, Subhash Suri
UC Santa Barbara, USA

Saket Saurabh
IMSc Chennai, India
UIB Bergen, Norway

Problem Definition



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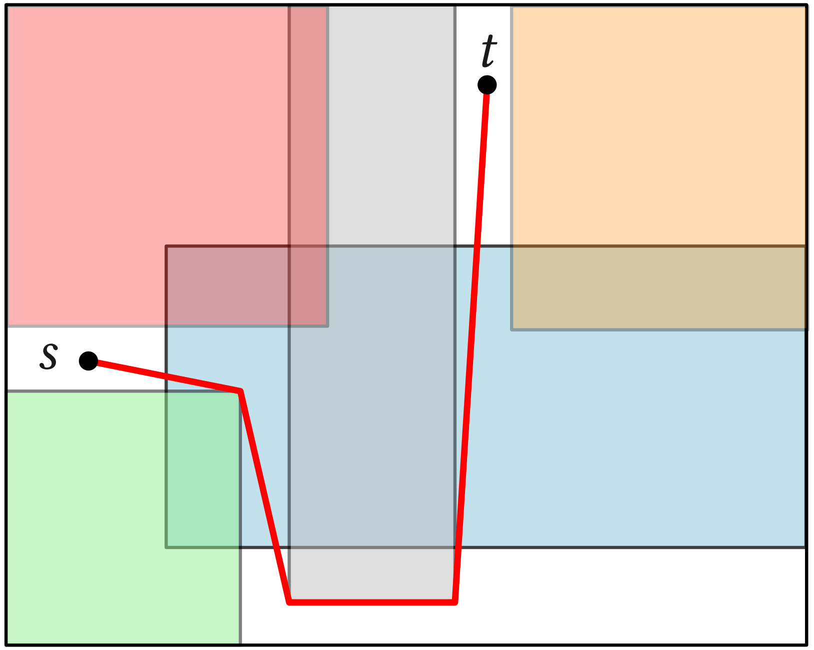


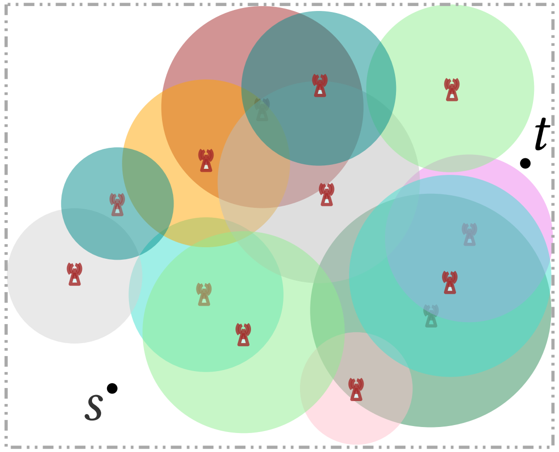
Input : An arrangement of obstacles in plane, source s, target t

Find an s–t path that intersects a minimum number of obstacles

Problem History

- *Barrier Resillience* in sensor networks
  – disk obstacles

Resillience of sensor network to an adversary



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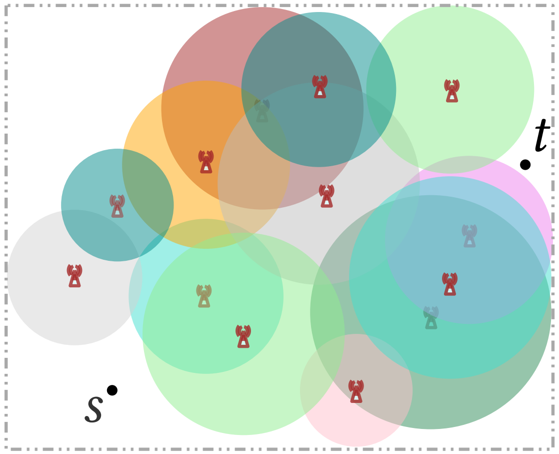

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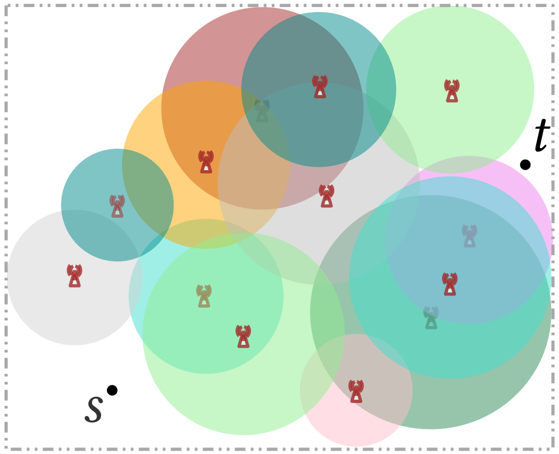

Resillience of sensor network to an adversary

- *Minimum Constraint Removal* in robotics and computational geometry

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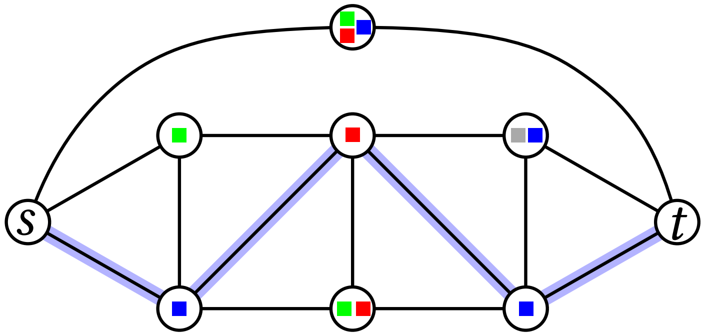
- *Min-Color Path* in graph theory

  – Vertices assigned a subset of colors $\{1, 2, \ldots, m\}$ as $\sigma : V \to 2^m$

  – find an s-t path π with minimum colors

minimize colors in $\sigma(\pi) = \bigcup_{v \in \pi} \sigma(v)$



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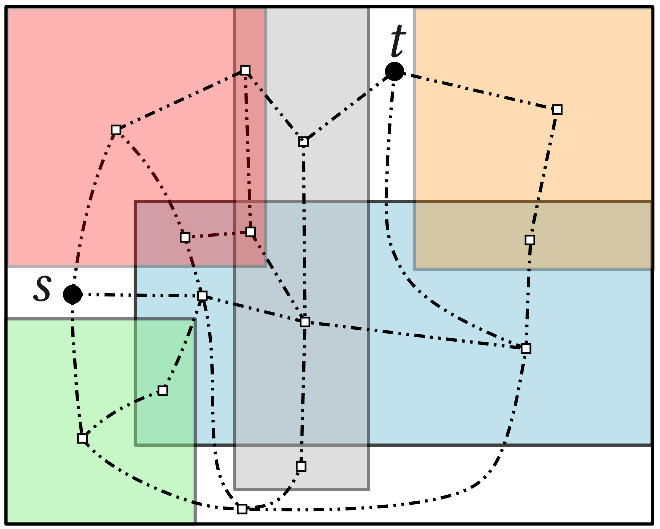
- build a graph over the arrangement
- obstacles correspond to colors
- to each vertex, assign colors incident to it



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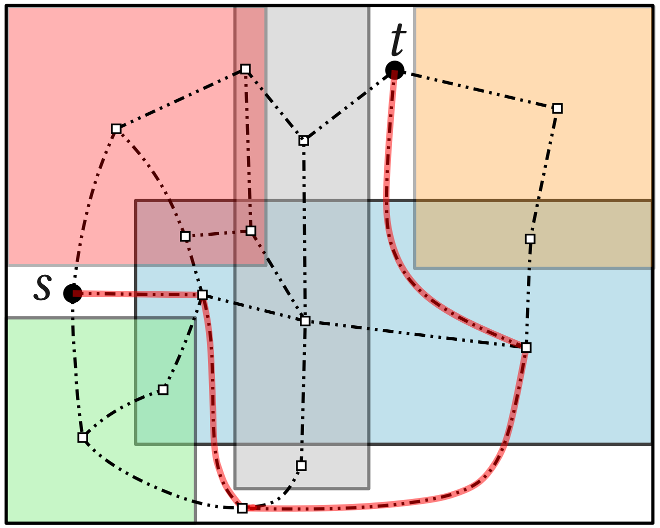

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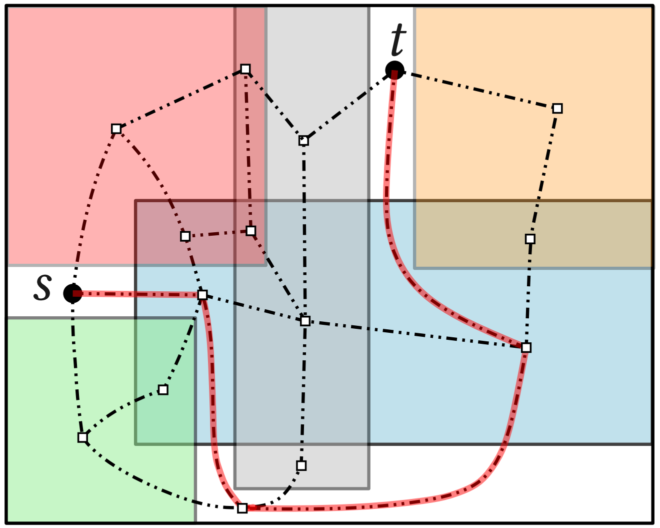

This graph is planar and color-connected (vertices containing any given color are connected)

Our Results

- Constant factor approximation algorithm for **Min-Color Path** on **color-connected** and **planar** graphs

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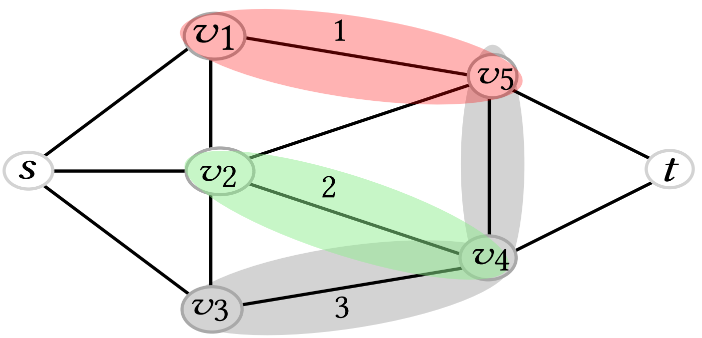
Given color-connected planar graph (G, σ), find a min-color path



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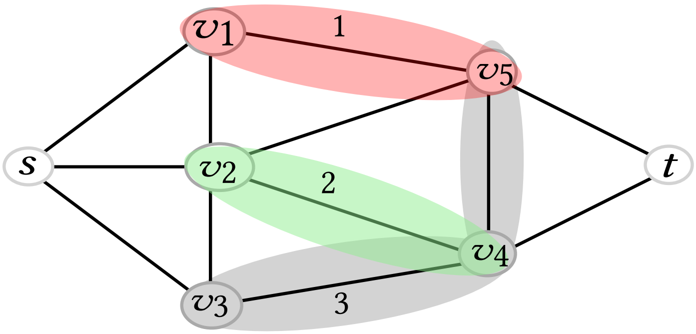

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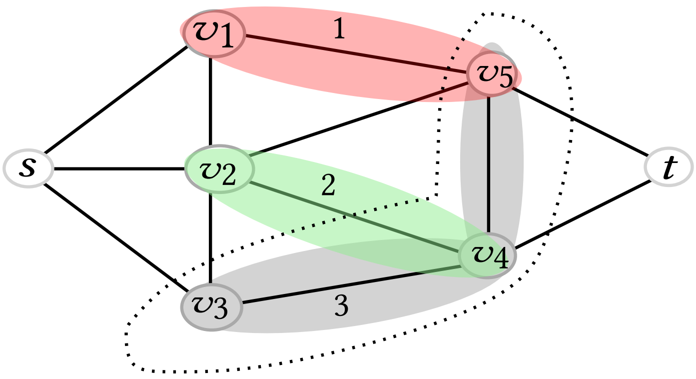

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Color Separator : set of colors S

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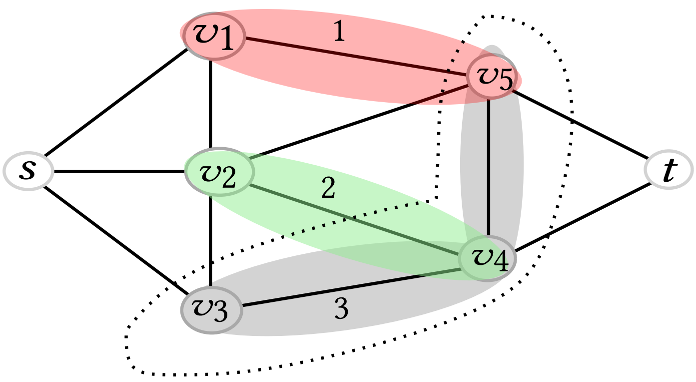

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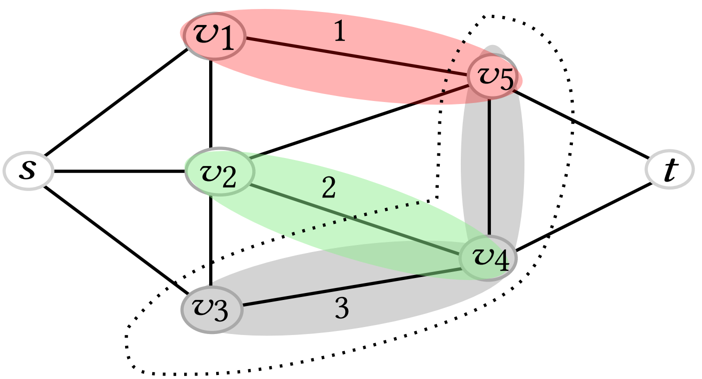

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smallest $\mathcal{C}^* \subseteq \{1, \ldots, m\}$ such that $\mathcal{C}^* \cap S \neq \emptyset$ for every $S \in \mathcal{F}$

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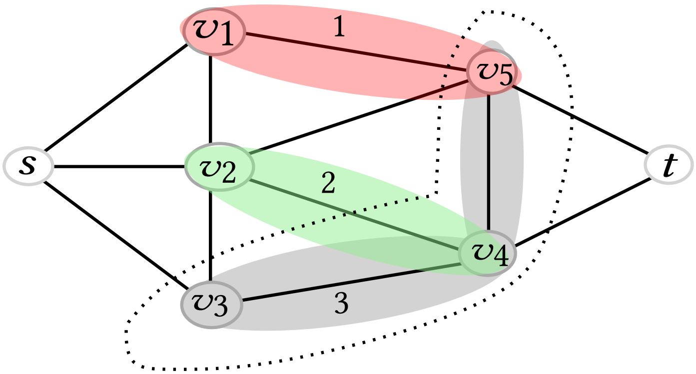

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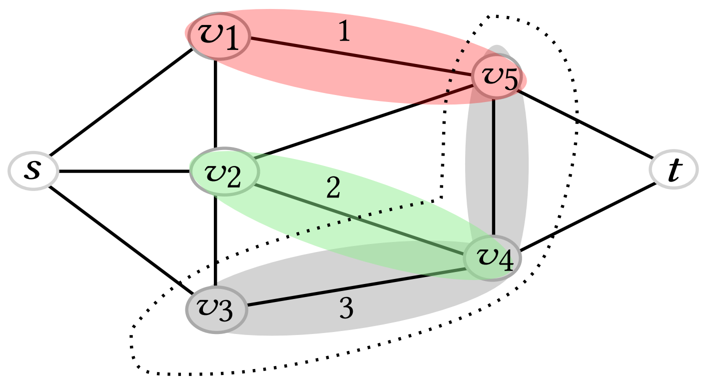

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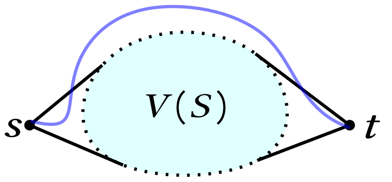

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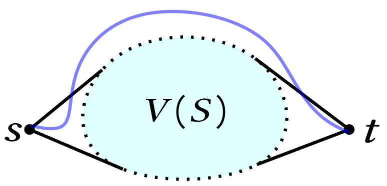

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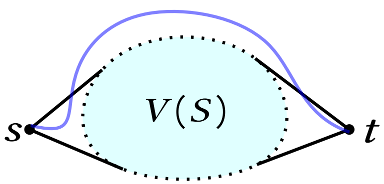

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- Remove colors in $\mathcal{C}^*$ from G, that is, $\sigma'(v) = \sigma(v) \setminus \mathcal{C}^*$
- (G, σ') contains a path π' with zero colors

because if not $S' = \bigcup_{v \in V} \sigma'(v)$ will be a color separator with $S' \cap \mathcal{C}^* = \emptyset$ contradiction

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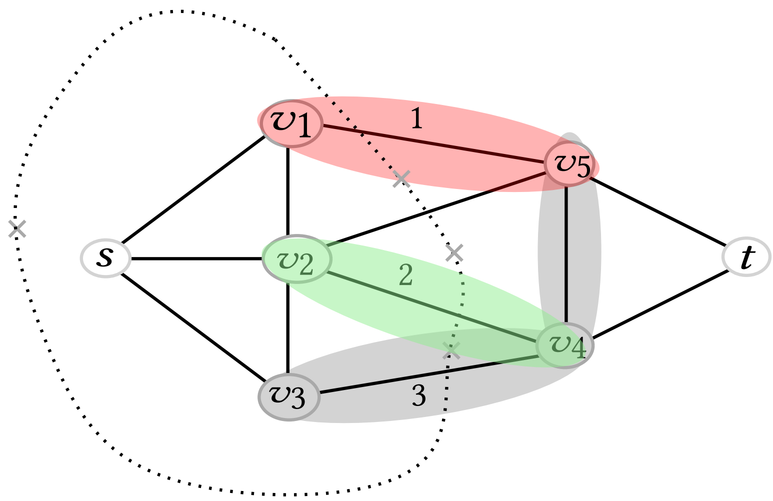
round x_i values to obtain an integral solution $\hat{y} = \{y_1, y_2, \ldots, y_m\}$

Lemma: Exists a rounding algorithm such that $\sum y_i = O(1) \cdot OPT$

Properties of Color Separators (on color-connected planar graphs)

Color Separator S corresponds to a **separating cycle** γ in dual graph G^*



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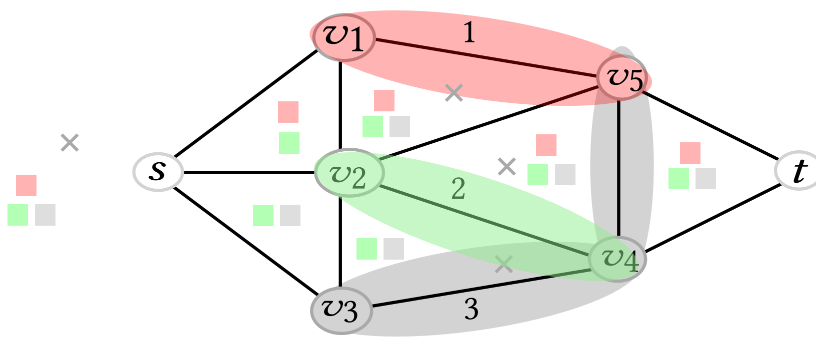

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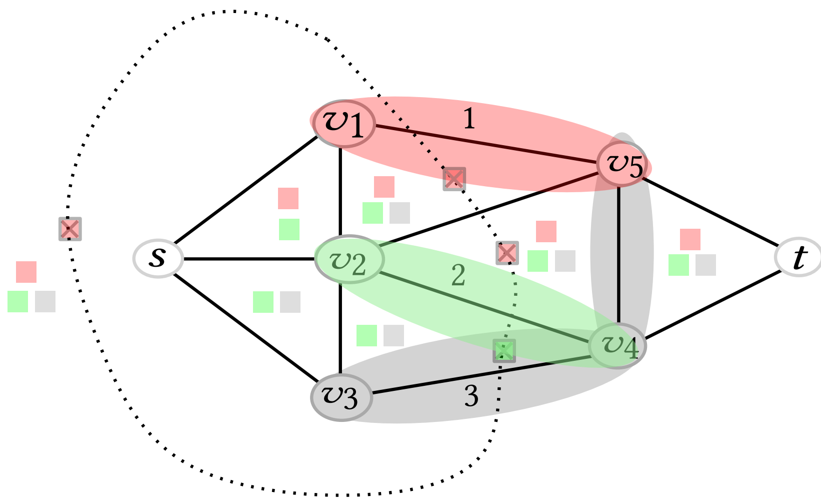

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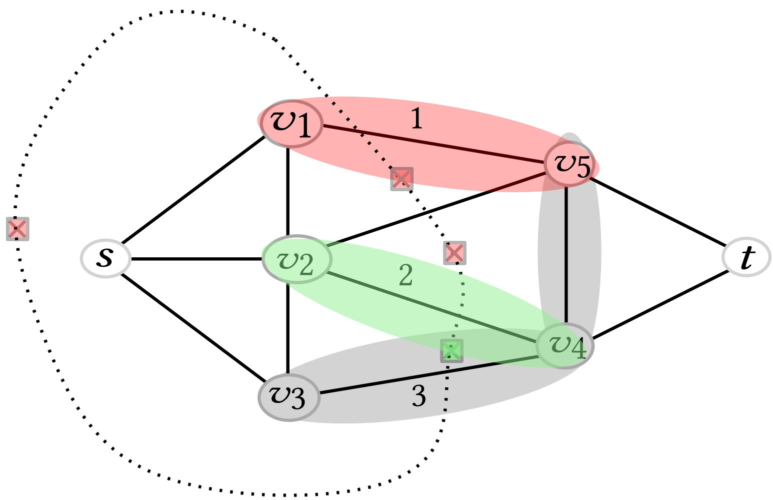

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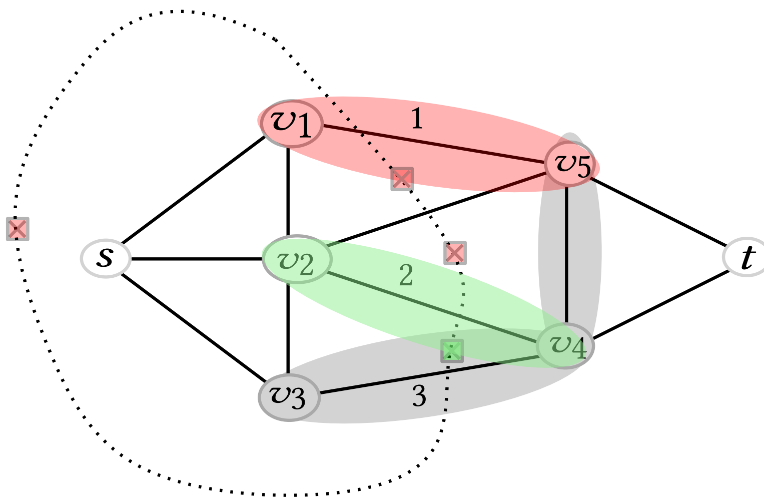

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Color Separators ≡ sequence of overlapping colors

Rounding Algorithm Overview

- Build a *color-intersection* graph $\mathcal{G}$
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  - Use the **small diameter decomposition** for such graphs by Lee'17
  - gives a set of vertices X such that $|X| = O(1) \cdot \sum d(c_i)$ and diameter of $G - X$ is at most 0.4
- Return $\mathcal{C}^*$: the set of colors corresponding to vertices in X

Lemma: $|\mathcal{C}^*| = O(1) \cdot OPT$ and the color set $\mathcal{C}^*$ hits all color separators

Rounding Algorithm Overview

- Build a *color-intersection* graph $\mathcal{G}$
  - contains a vertex c_i for every color i
  - contains edge (c_i, c_j) if colors $\{i, j\} \in \sigma(v^*)$ for some dual vertex v^*
- Assign *distance* value $d(c_i) = x_i$ to each vertex of $\mathcal{G}$
- Graph $\mathcal{G}$ is node-weighted *region intersection graph*
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uses that diameter of $G - X$ is small

In Summary

- $O(1)$-approximation for Min-Color Path on color-connected planar graphs
  - Equivalent to hitting all color separators with fewest colors
  - LP formulation that is polytime solvable on color-connected planar graphs
  - Round using small diameter decomposition of a color-intersection graph

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Thanks!